LIMITING MODE OF THE KINETIC EQUATION FOR A REACTOR

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The conditions are obtained for ensuring arrival at the limiting mode of a reactor, entering a region of periodic, almost periodic or time-limited external perturbation sources.

Consideration of the neutron balance in an element of phase space leads to the necessity for investigating the Boltzmann transport equation [1, 2]

$$\frac{\partial N(\mathbf{r}, \mathbf{v}, t)}{\partial t} = -\frac{\mathbf{v}}{v} \nabla\Phi(\mathbf{r}, \mathbf{v}, t) - \Sigma_a(\mathbf{r}, v)\,\Phi(\mathbf{r}, \mathbf{v}, t) + \int \Sigma_s(\mathbf{r}, \mathbf{v}' - \mathbf{v}, t)\,\Phi(\mathbf{r}, \mathbf{v}', t)\,d\mathbf{v}' + \int (1-\beta)\,\chi(v)\,\nu\Sigma_f(\mathbf{r}, v')\,\Phi(\mathbf{r}, \mathbf{v}', t)\,d\mathbf{v}' + \sum_i \lambda_i C_i(\mathbf{r}, t)\,\chi_i(v) + S(\mathbf{r}, \mathbf{v}, t), \tag{1}$$

where $\Phi(\mathbf{r}, \mathbf{v}, t) = vN(\mathbf{r}, \mathbf{v}, t)$ is the directional flux density of neutrons having velocity $\mathbf{v}$ at the point $\mathbf{r}$ and at the instant t.

After averaging over the space variable, the equation for the kinetics of the reactor can be given the following form:

$$\frac{d\varphi(t)}{dt} T = \left[\frac{k_{\text{eff}}(t) - 1}{k_{\text{eff}}(t)} - \sum_i \beta_i\right] \varphi(t) + \sum_i \beta_i \lambda_i \int_{-\infty}^{t} \varphi(\tau) \exp\left[-\lambda_i (t-\tau)\right] d\tau. \tag{2}$$

In the first place, we shall be interested in the condition for ensuring arrival at the limiting mode of a reactor, moving space and entering a region of external perturbing sources q(t), which are periodic or almost periodic in time and which may be also of a control nature. In this case, we do not exclude possible perturbations in the kernel under the integral sign in Eq. (2).

We shall say that the continuous curve $h = h(t)$ is the limiting mode of the family of continuous curves $\Gamma\{u(t)\}$, if $|u(t)-h(t)| \to 0$ when $t \to \infty$.

We shall call the function h(t) the m-function, if it belongs to one of the following classes of continuous functions:

1. periodic, with period σ;
2. periodic, with period 2σ and changing signs during the half-period ($h(t + \sigma) = -h(t)$);
3. bounded by a numerical straight line;
4. almost periodic according to Bohr, i.e., for the continuous function h(t), $-\infty < t < +\infty$, a value of $l = l(\varepsilon) > 0$ corresponds to every value of $\varepsilon > 0$ such that in any interval $[t_0, t_0 + l(\varepsilon)]$ there is at least one value τ for which $|h(t)-h(t+\tau)| < \varepsilon\,(-\infty < t < +\infty)$. The simplest example of almost periodic function (pp-functions) are periodic functions. Trigonometric functions

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$$x(t) = \sum_{i=1}^{n} (a_i \cos \lambda_i t + b_i \sin \lambda_i t) \quad (*)$$

have near-periodicity.

We note, finally, the following: a) trigonometric polynomials (*) with incommesurable λ_i do not possess the property of periodicity; b) differentiation and integration operations, generally speaking, derive from a class of pp-functions which naturally complicate their investigation. On the other hand, suppose that a pp-linear differential operator L is regular, i.e. it provides uniqueness of the solution $x(t) \in C^m(R^n)$ of the equation

$$Lx \equiv \sum_{i=0}^{m} A_i(t) \frac{d^i x}{dt^i} = f(t)$$

for any right-hand side of $f(t) \in C(R^n)$. Then the property of the pp-operator L is regularly stable relative to small perturbations $A_i(t)$.

We note that the set of functions (*) is dense in the space of the pp-function $E(\tilde{R})$, i.e. every pp-function can be approximated with any degree of accuracy by a trigonometric polynomial (Bohr).

Let us consider the autonomous vector system

$$\frac{dx}{dt} = F(x), \quad (**)$$

where $x = (x_1, \ldots, x_n)$, $F = (F_1, \ldots, F_n)$ and $\partial F_j/\partial x_n \subset C(R)$, i, h = 1, . . . , n. For every $t \in (t_1, t_2)$ there exists a neighborhood U of the point x_0 such that, for any $u \in U$ the solution x(t, u, 0) defines the mapping T or T_t of the neighborhood U in R^n. For these mappings, the relation $T_t T_s = T_{t+s}$ occurs or, generalizing, we have a single-parameter group in topological space. If we consider the open set $V_0 \subset R^n$ and the corresponding solution x(t, u, 0), $u \in V_0$, then for every $t \in (t_1, t_2)$ the mapping $T = T_t$ converts V_0 into the open set V_t. We denote by mes V the Lebesgue (volume) measure of the open set $V \subset R^n$. The following statements are well-known: Statement 1: if div F = 0, then mes V_t = mes V_0 for every $t \in (t_1, t_2)$ and the transform T_t conserves the measure.

Statement 2: if div F = 0, and M is an invariant set of an autonomous system and v_0 is part of M, being a Borel set of positive measure mes $|v_0| > 0$, then there exists a sequence t_m, m = 1, 2, . . . , such that $t_m \to +\infty$ when $m \to +\infty$ ($t_m \to -\infty$) and $v_0 v t_m \neq 0$, m = 1, 2, ...

If we consider v_0 as a small neighborhood of the initial point x_0 then, following Poincaré, statement 2 can be interpreted as: with probability equal to unity, the random motion is infinitely often reverting to its initial state. This property of mechanical systems is called stable according to Poisson (more frequently, this state is called recurrent motion).

Statement 1, by means of hydrodynamic analogy, is used for the characteristics of incompressible motion or flow (ergodic theory).

The properties of recurrence of the solutions of dynamic systems suggest: will these solutions be periodic or near-periodic? Although generally speaking, this is not true, however, the theorem is valid (C. R. Rutnam and L. L. Helms).

If div F = 0 and if the solution x(t) of the system (**) exists on $-\infty < t < +\infty$ and it is bounded and stable according to Lyapunov on both sides, then this solution x(t) is near-periodic according to Bohr.

The results relative to periodic and pp-motions are given in [4-6]. What has been said above indicates that the class of pp-functions is quite extensive and interesting from the physical point of view.

Let us describe our result.

We denote by $V(t, \tau)$ the normal operator solution, when $t = \tau$ of the equation

$$\frac{dv(t)}{dt} = A_1 v(t) + \frac{1}{T} \sum_i \beta_i \lambda_i \int_{\tau}^{t} \exp\left[-\lambda_i (t-s)\right] v(s)\, ds, \quad (3)$$

where

$$A_1 \overset{\text{def}}{=} \frac{1}{T}\left(1 - \sum_i \beta_i\right).$$

Together with Eq. (2) we shall consider the equation

$$\frac{d\varphi(t)}{dt} = \left[A_1 - \frac{1}{Tk_{\text{eff}}(t)}\right]\varphi(t) + \int_{-\infty}^{t}\left\{\frac{1}{T}\sum_i \beta_i\lambda_i \exp\left[-\lambda_i(t-s)\right] + M(t, s)\right\}\varphi(s)\,ds + q(t), \tag{4}$$

where M(t, s) is a certain kernel, obtained by perturbation of Eq. (2).

We shall assume that:

1) the real parts of all roots of the characteristic polynomial of the equation

$$\frac{du(t)}{dt} = A_1 u(t) + \frac{1}{T}\sum_i \beta_i\lambda_i \int_{-\infty}^{t} \exp\left[-\lambda_i(t-s)\right] u(s)\,ds \tag{5}$$

are negative and, consequently, the inequality

$$|V(t, \tau)| \leqslant B\exp\left[-k(t-\tau)\right]$$

is fulfilled, where the quantities B and k are independent of t and τ.

2) q(t) is an m-function

$$3)\ \int_{-\infty}^{+\infty}\left\{\left|\frac{1}{k_{\text{eff}}(t)}\right| + \int_{-\infty}^{t}|M(t, s)|\exp\left[-k(t-s)\right]ds\right\}dt < +\infty.$$

Then all solutions of Eq. (4) have one and the same limiting m-mode.*

Note. The determination of the characteristic polynomial for equations of the type (5) is given in [3]. In our case, this algebraic equation has the form

$$x - A_1 - \frac{1}{T}\sum_i \frac{\beta_i\lambda_i}{x+\lambda_i} = 0.$$

NOTATION

T	is the average lifetime of a neutron in the reactor;
$k_{\text{eff}}(t)$	is the effective neutron multiplication factor in the reactor;
β_i	is the effective fraction of the i-th group of delayed neutrons, taking account of their importance;
λ_i	is the decay constant of the i-th group;
q(t)	is the effective strength of added sources, taking account of their importance;
$\tilde{R}$	is the metric space with the metric $\rho(x, y)$;
R^n	is the n-dimensional Euclidean space;
$C(\tilde{R})$	is the space of functions, continuous and bounded by $(-\infty, \infty)$;
$C^m(R^n)$	is the space of m-times continuously differentiated functions with values in R^n;
$B(\tilde{R})$	is the space of pp-functions with the metric $\rho(x, y) = \sup_{-\infty<t<\infty} \rho[x(t), y(t)]$, $(x, y \in C(\tilde{R})$.

*A limiting m-mode denotes that the limiting mode is an m-function.

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